

Estimating Implied Correlations for Currency Basket Options Using the Maximum Entropy Method

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Introduction

Over the past few years, many large corporations have become more interested in hedging their currency exposures using basket options [Bensman]. Consider a US company that has manufacturing operations in Latin America. The company pays for its operations throughout the year but expects to sell only at the end of the year (e.g. Christmas time). The company's expenses are mostly in Latin American currencies but its revenues are in US dollars. If these currencies were to appreciate against the US dollar, the company might lose money.

Traditionally, the company may have hedged each exposure independently. For example, to hedge its Mexican peso exposure, the company might buy a US dollar put / Mexican peso call whose payout is given by $N * \max(c(T) - X, 0)$ where N is the notional amount, $c(T)$ is the US dollar / Mexican peso currency rate at expiration date T and X is the strike. Suppose that the spot rate is 0.109 (i.e., we need \$0.109 US dollars to purchase one Mexican peso). If the company buys an at-the-money option and the Mexican peso currency appreciates to a rate of 0.25, say, the company will be paid upon expiration of the option.

Companies that are exposed to a variety of currency fluctuations may find it preferable to directly hedge their aggregate risk using a basket option. The company can typically purchase an option on a basket of currencies at a lower price than it can purchase a combination of many separate options on the individual currencies.

It is clear that the price of the basket option is strongly dependent upon the correlation between the exchange rates. The lower the correlation between the currencies, the lower the volatility of the basket, which in turn lowers the fair value of the basket option. Lower correlations will cause movements in the individual currencies to offset each other, at least somewhat. Thus, an option struck on the basket should decrease in value as the pairwise correlation numbers decrease. We can construct a matrix of these correlations and analyze the sensitivity of the basket option value to changes in the matrix.

Although the value of the basket option certainly depends upon correlation matrix, it is not clear what the entries in this matrix should be. One approach is to use historical return correlations. In this approach we need to determine how to weight past returns and how many returns should be used. More significantly, the option value should depend upon predicted future correlations. These are the correlations that will actually be observed during the life of the option.

Financial practitioners often tend to think more about trading implied quantities than option prices. For example, currency option market makers typically quote implied volatility numbers rather than dollar values. Similarly, basket option traders tend to think in terms of "implied correlation". While implied volatilities can be uniquely determined from the price of an option on a single asset, it is generally not possible to uniquely specify the implied correlation numbers between assets given the price of a basket option.

In special circumstances, the implied correlation between currencies can be uniquely determined using “currency triangles”. For example, the market may provide implied volatility quotes on the US dollar / Japanese yen, European euro / Japanese yen and US dollar / European euro pairs (the three pairs are together called a “triangle”). We can then explicitly solve for the correlation between the euro and yen, denominated in US \$.)

However, in the general case, we are faced with a basket option on currency pairs without tradable options. To our knowledge, there is no liquid market for options on Mexican peso / Brazil real. In this case, many correlation matrices will generate the same market price of a given basket option and we are faced with the problem of choosing among them.

In this paper, we shall use entropy as a means of choosing the correlation matrix which

- a. Matches the implied volatility of the basket option.
- b. Resembles the historical correlation matrix and
- c. Makes as few other assumptions as possible.

In this way, we are able to give a more precise meaning to the notion of “implied correlation”.

Using our method, it is possible to trade correlations in nearly the same way as volatilities. For example, if the implied correlation matrix for a basket option is much higher than the historical correlation matrix over any time window, it may make sense to sell the basket and hedge using options on the individual currencies.

Financial Applications of the Maximum Entropy Method

Recently, several papers have appeared which use the maximum entropy method (MEM) as a way to value options. [Zou] introduce the concept of a “risk-neutralized historical distribution” p^* which generates the forward price of a stock. If we set $S(t)$ to be the spot price, $F(t, T)$ the forward price of S (observed at t and with maturity date T) and r the risk-free rate, then $F(t, T) = S(t)e^{r(T-t)}$. In general, there are many distributions p matching the forward price, so [Zou] use the notion of entropy to select the “most likely” one. The idea is to choose p as close as possible to the historical return distribution for S (according to some notion of distance) while still matching the forward price of the stock. After the “best distribution”, p^* , is determined, it is then possible to value any option on the stock with maturity date T by calculating its discounted expectation relative to p^* . For example, if $C(K, T)$ is a call option with strike K and maturity time T then we can value C according to $C(K, T) = e^{-r(T-t)} E(\max(S(T) - X, 0) | S(t), p^*)$. (Thus, the value of the call option is the discounted expected payoff of the call at maturity, given the probability density function p^* .) Using these calculated prices, we can then build an implied volatility skew for the stock S .

More generally, if some options along the skew are more liquid than others, p^* can be chosen to match liquid option prices as well as F and then used to price other options (such as exotics) more accurately. This idea was developed in [Hawkins] and generates a distribution which is not only consistent with the forward but also with the option volatility skew. Using a similar

approach, [Avellaneda] constrains p^* to match forward rate agreement (FRA) prices and interest rate swaps.

In general, MEM is useful in any context where we need to specify an implied quantity from a variety of alternatives. We shall use MEM to determine the “most likely” matrix of exchange rate correlations using observable option prices as constraints.

Description of the Maximum Entropy Method

The entropy of a probability distribution is a measure of the uncertainty of the distribution. Generally speaking, the entropy is maximal for a uniform distribution. It is minimal when one event has probability 1 and all other events have probability 0 of occurring.

In the above references, the relative entropy of a probability distribution q given a distribution p is defined: $S(p, q) = -\sum q_i \ln(\frac{q_i}{p_i})$. In this case, $S(p, q)$ is maximized when $p_i = q_i$ for all i . p is called the “prior” and would typically be a log normal distribution, although in general it could be any distribution. In [Zou], p is chosen to be the historical stock price return distribution, so that the risk neutralized distribution q is conditioned upon the past. Formally, the following problem is solved to find q :

- maximize $S(p, q)$ subject to the constraint $S(t)e^{r(T-t)} = E[S(T) | S(t), q]$.

We can proceed in a similar way to find the implied correlation matrix ρ^* given a basket option and options on each of the underlying currencies. In our case, ρ^* plays the role of q and the historical correlation matrix ρ_h plays the role of p . We describe our method in the next section.

We need to make a few technical modifications when we find an implied correlation matrix using MEM. For example, a correlation matrix can have negative values (so that the natural logarithm is not well-defined). Furthermore, the sum of entries in an $N \times N$ correlation matrix can range from $-N^2$ to N^2 , whereas $\sum p_i = 1$. We will deal with these issues in the next section.

Determining the Implied Correlation Matrix for a Basket FX Option

Here we expand upon the example given in the introduction. Suppose that a manufacturing firm has operations in Mexico, Brazil, Colombia and Peru. The firm pays for its operations throughout the year in local currencies. At the end of the year (e.g. Christmas season) it will sell the items produced in the US and receive US dollars. In the meantime, the manufacturer is exposed to exchange rate risk. If the US dollar were to weaken compared to the local currencies, the importer would not have enough local currency to cover its expenses. The manufacturer chooses to hedge with a basket option.

The basket option provides insurance against adverse movements in the basket of local currencies. For example, a European-style call with strike K might depend upon the weighted

average $c(t) = \sum_{i=1}^4 \alpha_i c_i(t)$ so that the payout at time T would be $\max(\sum_{i=1}^4 \alpha_i c_i(T) - K, 0)$.

(Here the α_i are the basket weights and c_1, c_2, c_3, c_4 are the \$ / Mexican peso, \$ / Brazilian real, \$ / Colombian peso and \$ / Peruvian new sol exchange rates, respectively.)

A portfolio equation relates the historical volatility σ of c to the historical volatilities σ_i of the

$$c_i: \sigma = \sqrt{\sum_{i=1}^4 \sum_{j=1}^4 w_i w_j \sigma_i \sigma_j \rho_{ij}} . \text{ Here } w_i = \frac{\alpha_i c_i}{\sum_{k=1}^4 \alpha_k c_k} \text{ and } \rho_{ij} \text{ is the historical correlation between}$$

the returns of c_i and c_j .

Option traders often replace the historical volatilities σ_i, σ with implied volatilities σ_i^*, σ^* in the above equation. The σ_i^* are determined from option prices on the currency pairs (e.g., options on the \$ / peso rate). Similarly, the implied volatility σ of $c(t) = \sum_{i=1}^4 \alpha_i c_i(t)$ can be derived from

the price of the basket option. In addition, the spot rates c_i in the portfolio equation are known. The correlation matrix ρ^* is the only unknown, and we need to choose ρ^* so that the equation is satisfied. Since ρ^* is an implied quantity for a given set of exchange rates and implied volatilities (assuming that the portfolio equation holds), we say that ρ^* is an ‘‘implied correlation’’ matrix. When ρ^* is very different from the historical correlation matrix ρ_h , there may be a trading opportunity. For example, if implied correlations are much higher than historical correlations, the basket option may be overvalued. In this case, it makes sense to sell the basket and to buy an appropriate combination of options on the individual currencies.

Calculating ρ^* is more difficult than calculating an implied volatility, since many correlation matrices will satisfy the portfolio equation when the portfolio contains more than two assets. By contrast, an implied volatility is uniquely defined from an option price using the Black Scholes equation. Thus, we need a rigorous way to specify the implied correlations. We can use the maximum entropy method as a way to choose ρ^* while making as few assumptions as possible. Proceeding in a similar way to [Zou], we can define the entropy of a matrix $\rho = \{\rho_{ij}\}$ as follows. For simplicity, we will assume that each $\rho_{ij} > 0$, so that the entropy is well defined. We next

normalize ρ so that its entries add up to 1: $\rho_{ij}^1 = \frac{\rho_{ij}}{\sum_{l,m=1}^4 \rho_{lm}}$. Then the entropy

$$S(\rho) = \sum_{i,j=1}^4 \rho_{ij}^1 \ln(\rho_{ij}^1) \text{ and } S \text{ is maximized when all of the matrix elements are equal. The}$$

entropy of ρ relative to the historical correlation matrix ρ_h can

be similarly defined: $S(\rho, \rho_h) = -\sum_{i,j=1}^4 \rho_{ij}^1 \ln\left(\frac{\rho_{ij}^1}{\rho_{(h)ij}^1}\right)$. In this case, $S(\rho, \rho_k)$ is maximized

when $\frac{\rho_{ij}}{\rho_{(h)ij}}$ is a constant. For any given ρ , there corresponds a volatility σ_ρ . Thus, we need to find $\rho = \rho^*$ which makes $S(\rho, \rho_k)$ as large as possible while keeping $|\sigma_\rho - \sigma|$ small. Since ρ is a correlation matrix, we also have the constraints $\rho_{ij} = 1$ for $i = j$ and $-1 \leq \rho_{ij} \leq 1$.¹ In the next section, we continue with our example and determine ρ^* explicitly.

Calculating the Implied Correlation Matrix ρ^*

In this section, we construct a concrete example where we can solve for ρ^* explicitly. In Table 1 we give the spot prices of various currencies and their weights in our sample basket. Note that the weights depend on the specific firm exposures.

	Bloomberg Ticker	\$ / currency	Sample Weight
Mexican peso	MXN	0.109	10
Brazilian real	BRL	0.400	2.5
Colombian peso	CLP	0.002	700
Peruvian new sol	PEN	0.286	3.5

Table 1. Value of one unit of foreign currency in US \$ (as of July 20, 2001)

For simplicity, we assume that the firm has approximately equal exposure in each of the foreign currencies. Thus, we have chosen different weights for each currency to ensure that a 1% change in the Colombian peso (which has a relatively small value) has roughly the same impact on the basket spot value as a 1% change in the real. Here, the value of the basket on July 20, 2001 would be equal to $10 * .109 + 2.5 * .4 + 700 * .002 + 3.5 * .286$, or 4.14. Table 2 gives historical return correlations over a two month historical window. We shall use the values in Table 2 as our prior.

	MXN	BRL	CLP	PEN
MXN	1.00	0.30	0.35	0.17
BRL	0.30	1.00	0.54	0.19
CLP	0.35	0.54	1.00	0.33
PEN	0.17	0.19	0.33	1.00

Table 2. Correlation Matrix for Historical Returns (May 24, 2001 to July 20, 2001)

¹ Formally, a real valued $n \times n$ correlation matrix C must satisfy the following conditions: C is symmetric; the diagonal entries of C are equal to 1; C is semidefinite (i.e., for any real valued column vector x of length n , $x^T C x \geq 0$). For simplicity, we have not constrained our matrix to be semidefinite in the analysis below.

In Table 3, we give sample at the money implied volatilities for options with 2 months to maturity. The implied volatility numbers given in Table 3 below are not necessarily accurate and are only given for illustrative purposes.

Currency	2 month ATM implied volatility
MXN	10%
BRL	25%
CLP	10%
PEN	8%

Table 3. Sample two month ATM implied volatilities

In order for our method to work, we need to assume that these values are reliable, so that the only unknown is the implied correlation matrix ρ^* . Assuming that the price of the basket call option is given and that the option has two months to maturity, we can calculate the implied volatility of the basket. (Here we have assumed that the exercise style is European, so that the Black Scholes formula applies.) We now consider two different scenarios.

Scenario A: the basket implied volatility is 11%

We can calculate the volatility of the basket using the portfolio equation and the implied volatilities given in Table 3; in this case, the basket should trade at a 10% implied volatility. However, since the short needs to hedge his position using options on the individual currencies (which incurs transaction costs and some risk), it is to be expected that the implied volatility will be somewhat higher than 10%. The implied correlation matrix given a market volatility of 11% accordingly has higher pairwise correlations than the historical matrix, as can be seen below:

	MXN	BRL	CLP	PEN
MXN	1.000	0.368	0.410	0.194
BRL	0.368	1.000	0.682	0.227
CLP	0.410	0.682	1.000	0.380
PEN	0.194	0.227	0.380	1.000

Table 4. Implied correlation matrix ρ^* in Scenario A

These values are relatively close to the historical correlations and we would not necessarily identify a trading opportunity.

Scenario B: the basket implied volatility is 13%

In this example, we assume that the basket option has an implied volatility of 13%. Again, we use MEM to obtain the implied correlation matrix shown in Table 5. Since the historical correlations are much lower than those in Table 5 below (over any historical window), it may be that the basket option is overpriced. Thus, it may be advantageous to sell the basket option and hedge volatility risk with a long position in individual currency options. This

trade would be done for a credit and would be relatively riskless (assuming the position can be efficiently delta hedged).

	MXN	BRL	CLP	PEN
MXN	1.000	0.926	0.855	0.366
BRL	0.926	1.000	1.000	0.509
CLP	0.855	1.000	1.000	0.745
PEN	0.366	0.509	0.745	1.000

Table 5. Implied correlation matrix ρ^* in Scenario B

Future Directions

In the above discussion, we have tacitly assumed that all options were approximately at-the-money. Our method can be generalized to incorporate options with different strikes. For example, it may be that, if the Mexican peso crashes relative to the US \$, correlations between Latin American currencies increase (i.e., everything crashes). In this case, implied correlations for out-of-the-money puts might be larger than at-the-money. (In other words, the volatility skew for each currency might not be steep enough to account for the basket implied volatility.) Given a set of liquid basket option prices with different strikes, it would be interesting to investigate changes in the implied correlation matrix as a function of strike.

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