SQUARE DEALS

Izzy Nelken presents a pricing methodology for complex choosers and compound options based on numerical quadrature

Unlike many other types of exotic options, whose pricing formulas consist of closed-form expressions, both complex choosers and compound options require a numerical method. Recently in *Risk*, Mark Rubinstein explained how to price chooser and compound options.[1] Rubinstein's method requires the solution of a non-linear equation.

The solution of non-linear equations typically involves the use of iterative methods. It is well known that there are cases where these methods do not converge or converge very slowly. Quadrature techniques are somewhat more robust and there may be an advantage to using them whenever possible. This article applies the non-arbitrage pricing methodology, which results in an integral. The integral is then computed using numerical quadrature techniques.

In an arbitrage-free world the price of any financial instrument is equal to the discounted expected value of all future payments received from that instrument. Following Rubinstein, we consider a European call option that, on expiry, will pay

$$\max(S_t - K, 0)$$

where K is the strike price and S_t is the spot price of the underlying at expiry, which is t years from now.

Today, however, we do not know S_t and therefore it is a random variable. Let the continuous-compounding risk-free interest rate be r. The price of the call option is

$$C = e^{-rt}E[\max(S_t - K, 0)]$$

where $E(A)$ is the expected value of random variable A. This can be written in integral notation as

$$C = e^{-rt}\int_0^\infty \max(S_t - K, 0)P(S_t)dS_t =$$

$$e^{-rt}\int_K^\infty (S_t - K)P(S_t)dS_t$$

where $P(S_t)$ is the probability that the spot price will be S_t at expiry.

Standard option pricing theory postulates that the spot price of the underlying follows a lognormal random walk. We assume that the spot price today is S_0, the volatility of the spot price is σ, the dividend rate q and we let z be a normally distributed random variable. Then

$$S_t = S_0 e^{\mu t + \sigma\sqrt{t}z}$$

where $\mu = (r - q - \sigma^2/2)$. With a change of variables we obtain

$$C = e^{-rt}\int_{\frac{\log(K/S_0) - \mu t}{\sigma\sqrt{t}}}^{\infty}\left[\left(S_0 e^{\mu t + \sigma\sqrt{t}z} - K\right)n(z)\right]dz$$

where $n(z)$ is the standard normal distribution density function

$$n(z) = \frac{1}{\sqrt{2\pi}}e^{\frac{-z^2}{2}}.$$

With some tedious calculus we can obtain a closed-form solution for the integral which is equivalent to the Black-Scholes pricing formula

$$C = e^{-qt}S_0 N(x) - e^{-rt}KN\left(x - \sigma\sqrt{t}\right)$$

with

$$x = \frac{\log(S_0/K) + rt - qt}{\sigma\sqrt{t}} + \frac{\sigma\sqrt{t}}{2}.$$

The Black-Scholes formula for a European put can be derived in a similar manner.

A chooser is an option which is neither a call nor a put. At a certain date, known as the "choice date", the holder of the chooser may trade it in for either a call or a put on the same underlying. If the call and the put have identical strikes and expiry dates, the chooser is known as a "regular chooser". If they differ in strike prices or expiry dates, the chooser is said to be "complex".

It is natural to assume that, on the choice date, the holder will trade the chooser in for the more expensive of the call and the put. Typically, if the spot price of the underlying on the choice date is low, the chooser will be exchanged for a put. On the other hand, if the price of the underlying is high, the chooser will probably be exchanged for a call.

Thus, on choice date, the value of the chooser is the maximum of the value of the call and the value of the put

$$\max\{C(S_t, K_c, T_c - t), P(S_t, K_p, T_p - t)\}$$

Here, the time to expiry of the call, the strike of the call, the time to expiry of the put and the strike of the put are denoted by T_c, K_c, T_p, and K_p respectively. S_t is the spot price on the choice date t years from now. $C(S,K,T)$ and $P(S,K,T)$ are the prices of European call and put options with strike price of K, time to expiry T, and the assumption that the spot price of the underlying is S. These may be calculated with the standard Black-Scholes formulas.[2] Other factors that have an impact on the price of a European option, such as the risk-free rate, the volatility and the dividend yield, have been omitted for clarity.

The main advantage of the chooser is that it is cheaper than a conventional straddle. Indeed, the price of the chooser depends on the choice date. The further away the choice date, the more flexibility the chooser has and the higher its price. Consider a simple chooser. If the choice date is today (and so $t = 0$), then the price of the chooser is the maximum of the put and the call. At the other extreme, if the choice date equals the expiry date ($t = $ [illegible]), the chooser becomes a straddle and its price equals the sum of the put and the call.

Rubinstein has shown that regular choosers can be priced via a closed-form formula. However, no known closed-form formula exists for pricing a complex chooser and a numerical method must be used

A compound option is an option on an option. That is to say, an option whose underlying itself is an option. There are four types of compound option: a call on a call, a call on a put, a put on a call and a put on a put.

For a compound option, we must distinguish between two expiry dates: the expiry of the compound option and the expiry of the underlying option. The time to expiry of the compound option is denoted by [illegible], and the time to expiry of the underlying

[1] Rubinstein, Mark, *Options for the undecided*, *Risk*, April 1991, page 43; *Double trouble*, *Risk*, December 1991–January 1992, page 73. See also Rubinstein, *Exotic options*, Finance working paper 20, Walter A Haas School of Business, Institute of Business and Economic Research, University of California at Berkeley, December 1991
[2] Black, F, and M Scholes, *The pricing of options and corporate liabilities*, Journal of Economics, May 1973, pages 637–659

[3] Geske, R, *The valuation of compound options*, Journal of Financial Economics, March 1979
[4] For more information about non-linear equations and their solution see Dahlquist, G. and A Björk, 1969, *Numerical methods*, Prentice-Hall

option by T; of course $t \le T$.

On the expiry date of the compound option, the payoff is determined by comparing the price of the underlying option with the strike price of the compound option. For example, the payoff for a call on a call is written as

$$\max\left[C(S_t, K1, T-t) - K2, 0\right]$$

where S_t is the spot price of the underlying the day the compound option expires, K1 is the strike price of the underlying option and K2 is the strike price of the compound option. Therefore, the quantity $C(S_t, K1, T-t)$ is the price of the underlying call option which still has $T - t$ years to expiry.

The original pricing formula for a call on a call was published by Geske[3] and was generalised by Rubinstein to cover all four cases.

Rubinstein's methods of pricing complex choosers and compound options are based on the solution of a non-linear equation. Non-linear equations are usually solved using iterative methods, which can be classified into two categories:

■ **Methods that guess an interval where the solution is found.** Given an interval that contains the root of a continuous function, the *bisection method* keeps dividing that interval until it becomes small enough. Once the interval collapses, we know where the root is. The bisection method is considered very robust and is guaranteed to converge. Its rate of convergence is considered slow, as it gains only a single bit of accuracy for each iteration. The *false position method* usually exhibits faster convergence but may at times result in stagnation.

■ **Methods that guess the solution.** Given a guess of the root, the *Newton-Raphson method* updates the guess and finds a closer estimate of the solution. The method is applied repeatedly until the estimate is close enough to the solution. This method has a quadratic order of convergence and, in general, converges much faster than the bisection method. However, there are cases, such as when the derivative of the function is close to zero at the initial guess, when the method will diverge. Similar comments apply to other methods, such as *Regula-Falsi* or the *Secant method*.

All methods that attempt to solve a non-linear equation require an initial guess and their behaviour is highly dependent on that guess. Furthermore, each of these methods exhibits slow convergence in certain cases.[4]

To avoid the pitfalls associated with solving a non-linear equation, we prefer to use numerical quadrature whenever possible. Another advantage is that quadrature algorithms do not require an initial guess.

Consider the complex chooser. According to the non-arbitrage pricing principle, we could write its price as

$$V = e^{-rt}\int_0^\infty \max$$

$$\left[C(S_t, K_c, T_c - t), P(S_t, K_p, T_p - t)\right] P(S_t)\, dS_t$$

As above, we make the substitution

$$S_t = S_0 e^{\mu t + \sigma\sqrt{t}z}.$$

A change of variables in the integral is also required

$$V = e^{-rt}\int_{-\infty}^{\infty} \max\Big[C\left(S_0 e^{\mu t + \sigma\sqrt{t}z}, K_c, T_c - t\right),$$

$$P\left(S_0 e^{\mu t + \sigma\sqrt{t}z}, K_p, T_p - t\right)\Big] n(z)\, dz$$

where μ and $n(z)$ are defined as before.

Turning to compound options, we describe the pricing of a call on a call. Other compound options are valued similarly.

According to the non-arbitrage methodology, we can write the price of the compound option as

$$O = e^{-rt}$$

$$\int_0^\infty \max\left[C(S_t, K_1, T-t) - K2, 0\right] P(S_t)\, dS_t$$

and, as before, perform a change of variable to obtain

$$O = e^{-rt}$$

$$\int_{-\infty}^{\infty} \max\left[C\left(S_0 e^{\mu t + \sigma\sqrt{t}z}, K_1, T - t\right) - K2, 0\right] n(z)\, dz.$$

In an implementation, we would prefer a finite range of integration. Fortunately, if the absolute value of z is large, the value of $n(z)$ is very small. Thus, we only need to integrate between $-M$ and M, where the value of M is determined by the required accuracy. In doing so we are "losing the tails" of the distribution. If these tails are small enough, the resulting loss in accuracy will be acceptable. Usually, setting $M = 6$ results in an approximation which is accurate to at least four decimal places, since we are computing everything within six standard deviations of the mean.

In principle, many quadrature methods could be used to evaluate the integrals above. Perhaps the simplest integration method is the left end point quadrature. Choose a large n and set $h = (b - a)/n$. Then we evaluate the function at n equally spaced points and approximate the integral as

$$\int_a^b f(x)\,dx = h\sum_{i=0}^{n-1} f(a + ih)$$

Other well-known methods are the trapezoidal method and Simpson's rule. The main drawback of this type of method is that the step size h is fixed throughout the integration. To get accurate results, h must be very small, which will result in many function evaluations and thus in an inefficient algorithm.

To implement the algorithms presented above, we need a numerical integrator that is quick, accurate and efficient. In particular, the functions that we are integrating are smooth in some areas and erratic in others. For example, if z is very large, $n(z)$ is very close to zero and we do not want to evaluate the function too many times in that region. On the other hand, at the points where the max( ) function changes behaviour, the function is not so smooth and we will probably need many function evaluations to get accurate results.

Some of the most commonly used numerical quadrature routines are based on an adaptive control strategy coupled with a local quadrature module. The adaptive control strategy divides the area of integration into two regions. At each region we evaluate the integral and approximate the error in the integration. This is done by using two different integration rules in each region. The error can be approximated using the difference between the two rules. If the error is small,

[5] See Kahaner, D, C Moler, and S Nash, 1989, *Numerical methods and software*, Prentice Hall Series in Computational Mathematics)
[6] The values of w_i and x_i are given in Kahaner, Moler and Nash (see note 5)

we assume that the function is smooth and that our approximation is valid. If the error is large, it is a sign that we need to do more work in that region.

One such adaptive routine is Q1DA,[5] which is based on the Gauss-Kronrod quadrature. This routine has been found to be quite reliable and efficient. We briefly describe the main ideas behind the routine.

The integrator is used to approximate the integral

$$I = \int_{x=a}^{b} f(x)\,dx$$

to a predetermined accuracy ε. It has two main parts: a local quadrature module (LQM) and an adaptive control strategy.

The LQM computes an approximation R and an error estimate E to the integral I such that $|I - R| \le E$.

The LQM in Q1DA uses the so-called Gauss-Kronrod seven–15-point formulas (G_7, K_{15}) to compute R and E. The advantage of the Gaussian rules is their high polynomial degree. Kronrod modified the Gaussian rules by adding an effective error control procedure. So, given an integration region and an integrand, the LQM computes G_7 and K_{15}. Both of these are given by rules of the type

$$\sum_{i=1}^{n} w_i f(x_i)$$

where the w_i are weights and the x_i are node points. The LQM simply evaluates the function at the points x_i, multiplies by the appropriate weights and sums the result.[6] K_{15} serves as an approximation to R, and using G_7 and K_{15} the LQM approximates E.

The adaptive control strategy evaluates an integral by dividing it into two parts $I = I_1 + I_2$ where

$$I_1 = \int_{x=a}^{(a+b)/2} f(x)\,dx$$

and

$$I_2 = \int_{x=(a+b)/2}^{b} f(x)\,dx$$

Using the LQM, Q1DA evaluates I_1 and I_2 and obtains the approximations R_1 and R_2 and the error estimates E_1 and E_2.

There are several possible outcomes

- If $E_1 + E_2 \le \varepsilon$ we accept the approximation and output the result $R = R_1 + R_2$.
- Otherwise, $E_1 + E_2 > \varepsilon$ and the approximation is not good enough. In this case we consider the region with the larger error estimate and re-integrate it with a modified error criteria.

For example, if $E_1 > E_2$ we will reconsider the integral I_1 and compute an approximation to it with an error bound of $\varepsilon - E_2$. This will be done by dividing it into two regions and proceeding as before.

- If both $E_1 > \varepsilon$ and $E_2 > \varepsilon$, we separately consider both regions. Each region is divided into two sub-regions and integration proceeds as before.

In areas where the function is smooth, the error estimates will be small and the approximations will be accepted. Areas where the function is not so well behaved will be divided again and again. Many function evaluations will be performed in such areas until the desired accuracy will be reached.

Thus, using a recursive approach the integral is computed quite easily. The advantage of an adaptive control strategy is that the integrator spends less time in regions where the function does not change much and more time in regions where the function swings wildly.

Consider a complex chooser being priced on January 1, 1993. Let the spot price of the underlying be $100, the risk-free rate 6%, the dividend rate 3% and the volatility 20%. The call has a strike of $110 and an expiry of December 1, 1993, and the put has a strike of $90 and an expiry of November 1, 1993.

The choice date is in the left column of the table and the price of the chooser is in the middle column. Note that as the choice date is pushed to the future, the option becomes more expensive.

If we were to change both expiry dates to December 1, 1993 and both strike prices to $100 we would obtain a regular chooser. If the choice date is January 1, 1993 the price is $8.70632 and if the choice date is December 1, 1993 the price is $14.7[illegible]809. We also consider a European call with a strike of $100 and an expiry date of December 1, 1993; its price is $8.70829. A similar put option is priced at $6.07177.

All this agrees with our intuition. If the choice date is today, the price of the chooser is the larger of the call and the put. On the other hand, if the choice date is the same as the expiry date, the price of the chooser is the sum of the put and the call. The small errors are the result of the rather tolerant integration error bounds.

We also consider a compound option being priced on January 1, 1993. It is a call option on a call option. As before, let the spot price of the underlying be $100, the risk-free rate be 6%, the dividend rate 3% and the volatility 20%. The strike of the underlying option is $100, its expiry is December 1, 1993, and the strike of the compound option is $5.

The left column of the table now represents the expiry date of the compound option; the right column the price of the compound option.

It is difficult to make significant timing measurements on a PC. A lot depends on factors such as the model of computer, clock speed, amount of RAM and which compiler processed the source code. On an IBM-compatible 486DX33 with 8MB of RAM, a cache of 256kB and a 128MB hard disk, computation of the first complex chooser in the table required 3.55 seconds. Pricing the first compound option took 6.13 seconds.

In sum, we have seen that quadrature-based techniques can be used in option pricing models to great advantage. They are both theoretically justifiable and computationally efficient. ■

Results

Date	Chooser	Compound
Feb 1, 1993	$5.18538	$3.85616
Mar 1, 1993	$5.58273	$4.11055
Apr 1, 1993	$5.93480	$4.39400
May 1, 1993	$6.21549	$4.65685
Jun 1, 1993	$6.45841	$4.91745
Jul 1, 1993	$6.65484	$5.16204
Aug 1, 1993	$6.82061	$5.41063

Izzy Nelken is president of Super Computer Consulting in Toronto. Ron Dembo and Algorithmics are gratefully acknowledged